B.Sc. IInd Year
(Maths Hons.)
Advanced Calculus
(Paper-IIIrd)

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Indeterminate Forms :-

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \frac{\lim_{x\to a} \phi(x)}{\lim_{x\to a} \psi(x)} \quad \text{(i.e. know)} \quad -(1)$$

If $\lim_{x\to a} \phi(x) = 0$ and $\lim_{x\to a} \psi(x) = 0$

$\Rightarrow$ From (1); $\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \frac{0}{0}$ which is seem to being meaningless. The form of $\frac{0}{0}$ is known as an indeterminate form. Other indeterminate form common use are. $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0.

Theorem or determination of limit of $\frac{0}{0}$ form :-

Let $\phi(x)$ and $\psi(x)$ are be two function which Taylor's expansion are given about $x=a$ and $\lim_{x\to a} \phi(x) = 0$ and also $\lim_{x\to a} \psi(x) = 0$

Then $\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)}$.

Proof :- Taylor's expansion of $\phi(x)$ and $\psi(x)$ about $x=a$ is given by

$$\phi(x) = \phi(a) + (x-a)\phi'(a) + \frac{(x-a)^2}{\lfloor 2}\phi''(a) + \cdots + \frac{(x-a)^{n-1}}{\lfloor n-1}\phi^{n-1}(a) + \frac{(x-a)^n}{\lfloor n}\phi^n\{a+\theta_1(x-a)\} \quad -(1)$$

$; 0<\theta_1<1$

$$\varphi(x) = \varphi(a) + \frac{(x-a)}{\underline{|1}}\varphi'(a) + \frac{(x-a)^2}{\underline{|2}}\varphi''(a) + \cdots$$

$$+ \frac{(x-a)^{n-1}}{\underline{|n-1}}\varphi^{n-1}(a) + \frac{(x-a)^n}{\underline{|n}}\varphi^n\{a+\theta_2(x-a)\},$$

where $0<\theta_2<1$

We have

$$\lim_{x\to a}\frac{\phi(x)}{\varphi(x)} = \lim_{x\to a}\frac{\phi(a)+\frac{(x-a)}{\underline{|1}}\phi'(a)+\frac{(x-a)^2}{\underline{|2}}\phi''(a)+\cdots+\frac{(x-a)^n}{\underline{|n}}\phi^n\{a+\theta_1(x-a)\}}{\varphi(a)+\frac{(x-a)}{\underline{|1}}\varphi'(a)+\frac{(x-a)^2}{\underline{|2}}\varphi''(a)+\cdots+\frac{(x-a)^n}{\underline{|n}}\varphi^n\{a+\theta_2(x-a)\}}.$$

We have $\phi(a)=0$ and $\varphi(a)=0$

$$\Rightarrow \lim_{x\to a}\frac{\phi(x)}{\varphi(x)} = \lim_{x\to a}\frac{(x-a)\phi'(a)+\frac{(x-a)^2}{\underline{|2}}\phi''(a)+\cdots+\frac{(x-a)^n}{\underline{|2}}\phi^n\{a+\theta_1(x-a)\}}{(x-a)\varphi'(a)+\frac{(x-a)^2}{\underline{|2}}\varphi''(a)+\cdots+\frac{(x-a)^n}{\underline{|n}}\varphi^n\{a+\theta_2(x-a)\}}.$$

$$= \frac{\phi'(a)}{\varphi'(a)}$$

$$\Rightarrow \lim_{x\to a}\frac{\phi(x)}{\varphi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\varphi'(x)}$$

Note :- (1) Here we need not be differentiated as quotient. We differentiate numerator & denominator separately.

Note :- (2). We have need to Ist check that it will be form of $\frac{0}{0}$.

Note 3: If $\phi(a) = \phi'(a) = \cdots = \phi^{n-1}(a) = 0$

& $\varphi(a) = \varphi'(a) = \cdots = \varphi^{n-1}(a) = 0$

and $\phi^{n}(a)$ and $\varphi^{n}(a)$ are not equal to 0 or $\pm\infty$ simultaneously.

We have $$\lim_{x\to a} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\phi^{n}(x)}{\varphi^{n}(x)}$$

Note 4: If $x \to \infty$ or $-\infty$; then we have

$x = \frac{1}{t} \to \infty$ or $-\infty$ as $t \to 0$

$$\lim_{x\to\infty} \frac{\phi(x)}{\varphi(x)} = \lim_{t\to 0} \frac{\phi\left(\frac{1}{t}\right)}{\varphi\left(\frac{1}{t}\right)}$$

$$= \lim_{t\to 0} \frac{\phi'\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)}{\varphi'\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)}$$

$$= \lim_{t\to 0} \frac{\phi'\left(\frac{1}{t}\right)}{\varphi'\left(\frac{1}{t}\right)}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\phi(x)}{\varphi(x)} = \lim_{x\to\infty} \frac{\phi'(x)}{\varphi'(x)}.$$

The determination of limit of indeterminate forms by this method is known as de-L'Hospital's Rule.

Example: Prove that $$\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2} = \frac{11}{24}e$$

Solution: let $y = (1+x)^{1/x}$

$$\Rightarrow \log y = \frac{1}{x}\log(1+x) = \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$$

$$\Rightarrow \log y = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

$$\Rightarrow y = e^{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots}$$

$$y = e \cdot e^{-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots}$$

$$\Rightarrow y = e\left[1 + \left\{-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right\} + \frac{1}{\lfloor 2}\left\{-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right\}^2 + \frac{1}{\lfloor 3}\left\{-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right\}^3 + \cdots\right]$$

$$\Rightarrow y = e - \frac{ex}{2} + \frac{ex^2}{3} + \frac{ex^2}{8} + \cdots \text{ the terms of higher powers of } x.$$

$$\Rightarrow (1+x)^{1/x} = e - \frac{ex}{2} + \frac{ex^2}{3} + \frac{ex^2}{8} + \cdots \text{ the terms of higher powers of } x.$$

$$\Rightarrow (1+x)^{1/x} - e + \frac{ex}{2} = \frac{11}{24}ex^2 + \cdots \text{ the terms of higher powers of } x$$

$$\Rightarrow \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \frac{11}{24}e + \cdots \text{ the terms of higher powers of } x.$$

($\frac{0}{0}$ form; either use L'Hospital's Rule or expansion method).

$$\Rightarrow \lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \lim_{x\to 0}\left\{\frac{11}{24}e + \cdots \text{ the terms of higher powers of } x.\right\}$$

$$= \frac{11}{24}e + 0$$

$$= \frac{11}{24}e \quad \text{(Hence proved)}$$

Exercise: Evaluate (i) $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$ (ii) $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$